Talk 3: DG Lie Algs

Recap Last "nonstandard" talk — tomorrow we begin w/ the course proper

Theme

dg Lie algebras $\longleftrightarrow$ deformation functors

Today we'll introduce & produce a functor $\longrightarrow$

This will raise some questions that we'll need to answer before we can give ~~the~~ full-blown account of the relationship

DG Lie Algebras & Deformation Functors

I Let $\mathbb{k}$ denote a characteristic zero field.

(For me, $\mathbb{k}$ is usually $\mathbb{C}$.)

Def A <u>differential graded Lie algebra</u> (or dgla) is

(a) a $\mathbb{Z}$-graded $\mathbb{k}$-vector space $(V^n)_{n\in\mathbb{Z}}$

(b) a degree +1 linear map d so a collection

$$(d^n : V^n \to V^{n+1})_{n\in\mathbb{Z}}$$ called the <u>differential</u>

(c) a bilinear map of degree zero

$$[\ ,\ ] : V^m \otimes V^n \longrightarrow V^{m+n}$$

$$x\otimes y \longmapsto [x,y]$$ [illegible]

satisfying

(1) $d^2 = 0$ (so (V,d) is a cochain cplx)

skew symmetry (2) $[y,x] = -(-1)^{xy}[x,y]$ ← xy as superscript just means degree

Jacobi identity (3) $[x[yz]] = [[xy]z] + (-1)^{xy}[y[xz]]$

Leibniz rule (4) $d([xy]) = [dx,y] + (-1)^{x}[x,dy]$

③ says $[x,-]$ acts as a derivation, which makes it easy to remember

Ex • Let V be an ordinary vector space, like $\mathbb{R}^n$

$\mathrm{End}(V)$ is a Lie algebra, as

there is no nontrivial differential possible,

with

$$[M,N] := M \circ N - N \circ M$$

Also known as $\mathfrak{gl}(V)$ ← for some reason, people use Fraktur for Lie algebras

• Let $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ be the matrices with trace zero. ← Exercise: why is bracket closed under this condition? Product isn't...

Rmk Believe it or not, you don't need to know any standard stuff about Lie algebras for this course — what I mean is that the stuff you'd learn in a rep theory class is not relevant (e.g., root systems, classification, etc).

//

• We can generalize the first example as follows.

Let (W, d_W) be a cochain cplx, or dg vector space.

Let

$$\mathrm{End}\, W = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(W,W)$$

where

$$\mathrm{Hom}^n(W,W) = \{ \varphi : W \to W : \varphi \text{ linear \& } \forall k \;\; \varphi(W^k) \subset W^{n+k} \}$$

We have a natural differential

$$d_{\mathrm{End}W}(f) = [d_W, f] = d_W \circ f - (-1)^f f \circ d_W$$

(why does $d^2_{\mathrm{End}W} = 0$?)

and bracket

$$[f, g] = f \circ g - (-1)^{fg} g \circ f .$$

I'll leave it to you to check the conditions, which all follow by commutator manipulations.

Rmk This kind of dg Lie algebra plays a central role, much like $\mathfrak{gl}_n$ in usual rep theory //

II Let's examine this dg Lie algebra a little.

An element $\varphi \in \mathrm{Hom}^1(W, W)$ is a degree 1 element & hence can be used to modify d_W

$$d_W \rightsquigarrow d_W + \varphi$$

← any other differential on $W^\#$ is of this form!

When is it a differential?

$$0 = (d_W + \varphi)^2 = d_W^2 + d_W \circ \varphi + \varphi \circ d_W + \varphi^2$$

$$= d_{\mathrm{End}W}(\varphi) + \tfrac{1}{2}[\varphi, \varphi]$$

This equation is an example of the Maurer-Cartan equation

Def Let $(V, d_V, [\,,\,])$ be a dg Lie algebra.
The Maurer–Cartan ~~element~~ map is

$$Q : V^1 \longrightarrow V^2$$
$$x \longmapsto d_V x + \tfrac{1}{2}[x,x]$$

A Maurer-Cartan element is $x \in Q^{-1}(0)$, i.e., it satisfies the MC equation

$$d_V x + \tfrac{1}{2}[x,x] = 0 .$$

The Maurer-Cartan locus is $\{x : Q(x) = 0\}$

~~Let's return to our example of End(W).~~

~~Note that for any $g \in \mathrm{Hom}^0(W,W)$,~~
~~Some changes of d_W are kind of "boring":~~
~~take an automorphism g of W~~
~~& conjugate to produce differential~~
~~$g \circ d_W \circ g$~~
~~We want to~~

Rmk: One can nearly always understand a deformation of something as a change in some linear map, hence the central role of MC eqn

Let's return to our example End(W)

Observe that any isomorphism of graded vector spaces

$$\begin{array}{ccccccccc}
\widetilde{W} = & \to & W^{-1} & \xrightarrow{d_{W,f}} & W^{0} & \xrightarrow{d_{W,f}} & W^{1} & \to \\
 & & \cong \downarrow f_{-1} & & \cong \downarrow f_{0} & & \cong \downarrow f_{1} & \\
W = & \to & W^{-1} & \xrightarrow[d_W]{} & W^{0} & \xrightarrow[d_W]{} & W^{1} & \to
\end{array}$$

induces a cochain isomorphism $d_W \rightsquigarrow d_{W,f} = f^{-1} \circ d_W \circ f$

And such f induces an isomorphism of MC loci:

$$\{\varphi : d_{\mathrm{End}\, W}\varphi + \tfrac{1}{2}[\varphi,\varphi]=0\} \xrightarrow{\cong} \{\tilde{\varphi} : d_{\mathrm{End}\, \widetilde{W}}\tilde{\varphi} + \tfrac{1}{2}[\tilde{\varphi},\tilde{\varphi}]=0\}$$

$$\varphi \longmapsto f^{-1} \circ \varphi \circ f$$

(Check if you'd like!)

Some isomorphisms are "self-generated" or "inner" and we will distinguish them

In a sense, we will "quotient them out"

Suppose now that each W^n is finite dimensional, only finitely many nonzero.

N.B. This benign-looking assumption actually excludes most of the "natural" examples. For example, $\mathrm{Hoch}^*(\mathrm{Sym}\, V)$.

We will circumvent this restriction in a moment

Then we have

$$\begin{array}{ccc} \mathrm{Hom}^0(W,W) \cong \prod_n \mathfrak{gl}(W^n) & \xrightarrow{\prod \exp} & \prod_n GL(W^n) \\ \downarrow \mathrm{ad} & & \\ \mathrm{Hom}^0(\mathrm{End}\,W, \mathrm{End}\,W) \cong \prod_n \mathfrak{gl}(\mathrm{Hom}^n(W,W)) & \xrightarrow{\prod \exp} & \prod_n GL(\mathrm{Hom}^n(W,W)) \end{array}$$

"adjoint action" (ad)

$a \longmapsto \exp(a) = \sum_n \frac{1}{n!} a^n$

$a \longmapsto \mathrm{ad}(a) = [a,-]$

$A \longmapsto \mathrm{Ad}A : \varphi \mapsto A^{-1}\varphi A$

finite-dim'l

In other words, every deg zero element $a \in \mathrm{Hom}^0(W,W)$ determines an automorphism $\exp(\mathrm{ad}(a))$ of $\mathrm{End}\,W$

<u>Fact</u> $\exp(\mathrm{ad}(a))(\varphi) = \underbrace{\exp(a)^{-1}}_{=\exp(-a)} \circ \varphi \circ \exp(a)$

Hence we can express the isomorphism <u>solely</u> in terms of the Lie algebra structure of $\mathrm{End}\,W$

In particular, as $\exp(x) = \mathrm{id} + x + \cdots$

we can see

$$\varphi \text{ s.t. } d_{\mathrm{End}}\varphi + \tfrac{1}{2}[\varphi,\varphi] = 0 \rightsquigarrow \tilde{\varphi} = \exp(-a)\circ\varphi\circ\exp(a) = \varphi + G_a(\varphi)$$

Explicitly (see Manetti)

$$G_a(\varphi) = \sum_{n=0}^{\infty} \frac{ad(a)^n}{(n+1)!}\left([a,\varphi] - d_W a\right).$$

Rmk This is called a "gauge action" & might feel somewhat familiar if you've messed around with connections on bundles. //

We can thus consider

$$\text{MC locus of End}W \Big/ \underset{\text{gauge action}}{\exp((\text{End}\,W)^0)}$$

as "deformations of differential on W modulo minor deformations"

III We will now use this approach to make a deformation functor out of a dg Lie algebra.

Fix $(V, d_V, [,])$ a dg Lie algebra.

For any commutative algebra $(A, \cdot)$,

$V\otimes A = \bigoplus_n V^n \otimes A$ is a dg Lie alg w/

$$d_{V\otimes A} = d_V \otimes id_A ,$$

$$[v\otimes a, v'\otimes a']_{V\otimes A} = [v,v']_V \otimes a\cdot a'$$

~~"dg Lie algebras are~~

Rmk A comm dg algebra ("cdga") is a graded comm alg w/ a differential compatible with product $d(aa') = da \cdot a' \pm a \cdot da'$.
A good example is the de Rham complex $(\Omega^*(X), d_{dR})$ of a smooth manifold. This construction also works for cdga's too! //

We thus have a functor

$$\mathrm{Art}_{\mathbb{C}} \xrightarrow{V \otimes -} \mathrm{dgLie}^{nil}$$

dgLie^{nil} = nilpotent dg Lie algebras

$\mathfrak{g}$ nilpotent if $\exists N \in \mathbb{N}$ s.t. $\forall x_1, \ldots, x_{N+1} \in \mathfrak{g}$, $[x_1,[x_2,[\cdots[x_N, x_{N+1}]\cdots]]] = 0$

$$\mathrm{Art}_{\mathbb{C}} \longrightarrow \mathrm{dgLie}_{\mathbb{C}}$$

$$A \longmapsto V \otimes m_A$$

note just max ideal!!

For nilpotent Lie algebras, all these formulas with exponentials make sense (only finitely many terms!) even if V is infinite-dimensional.

Hence we have

$$\mathrm{dgLie}^{nil} \xrightarrow{MC} \mathrm{Set}$$

$$\mathfrak{g} \longmapsto \{x \in \mathfrak{g}^1 : dx + \tfrac{1}{2}[x,x] = 0\}$$

and the gauge action is well-defined so we have

$$\mathrm{dgLie}^{nil} \xrightarrow{MC/gauge} \mathrm{Set}$$

$$\mathfrak{g} \longmapsto MC(\mathfrak{g}) / ~~gauge~~ ~~action~~ \exp(\mathfrak{g}^0)$$

Just use same formulas!

<u>Def</u> For $V \in \mathrm{dgLie}$, let

$$MC_V, \mathrm{Def}_V : \mathrm{Art} \longrightarrow \mathrm{Set},$$

denote the predeformation functor given by the composite

$$MC/gauge \circ (V \otimes -)$$

$$MC \circ (V \otimes -)$$

<u>Prop</u> Def_V is a <u>deformation functor</u>.

<u>Recall def</u>: For every fiber product in Art

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & A \end{array},$$

the canonical map

$$\mathrm{Def}_V(B \times_A C) \longrightarrow \mathrm{Def}_V(B) \times_{\mathrm{Def}_V(A)} \mathrm{Def}(C)$$

is (1) ~~a bijection~~ surjective when f surjective

(2) bijective when $A = \mathbb{K}$.

If

Let's check first for MC_V: ~~MC_V is a bijec~~

~~In this~~ in this case the canonical map is a bijection whenever f is surjective

Proof? Refer to Manetti

Computation: $T_{Def_V} \cong H^1(V)$

$$(V \otimes \varepsilon/\varepsilon^2)^1 \cong V^1 \xrightarrow{Q} (V \otimes \varepsilon/\varepsilon^2)^2 = V^2$$

$$\varepsilon x \longmapsto \varepsilon d_V x + \tfrac{1}{2}\varepsilon^2 [x, x]^{0}$$

$\to$ $MC_V(\mathbb{D}) = \ker(d : V^1 \to V^2)$

gauge action is simple:

$$\exp(ad(\varepsilon a)) = 1 + \varepsilon[a, -]$$

$a \in V^0$

so action is just

$$e^{\varepsilon a} * \varepsilon x = \varepsilon x + \varepsilon da + \varepsilon^2[a, x]^{0}$$

Ex 8.4

Claim: A universal obstruction theory for Def_V is $H^2(V)$ with ob such that

$$M \to B \twoheadrightarrow A$$

$$\Downarrow$$

$$Def_V(A) \xrightarrow{ob} M \otimes H^2(V)$$

$$x \in MC_V(A) \rightsquigarrow \tilde{x} \stackrel{\text{"}}{=} x + m' \in L^1 \otimes B \rightsquigarrow d\tilde{x} + \tfrac{1}{2}[\tilde{x}, \tilde{x}] \in V^2 \otimes M$$

Upshot

We have a functor

$$\text{dgLie} \longrightarrow \text{DefFun} \subseteq \text{Fun}(\text{Art}, \text{Set})$$
$$V \longmapsto \text{Def}_V$$

so we have established some kind of relationship between the two kinds of objects

Unfortunately, it loses a lot of information:

① Def_V only cares about V^0, V^1, V^2 and not the rest of V, which may be interesting! (cf. $\text{Hoch}^*(A)$!)

②, if $\varphi : V \longrightarrow W$ is a map of dg Lie algebras & a quasi-isomorphism, then

$$\text{Def}_V \cong \text{Def}_W$$

In other words, this functor is far from any sort of equivalence.

That might seem disappointing but in the 1990s, people instead interpreted it as sign that the target category of

deformation functors should be enlarged,
based on experience with important examples (like Koch π_1).
This was one of the prompts to develop global DAG.

<u>Anticipatory remark</u>: To deal with the issue of quasi-isomorphism, we will need to develop some higher category theory. Note that dgLie is a <u>non</u> linear category: you cannot (typically) <u>add</u> maps of Lie algebras! Thus ordinary homological algebra is insufficient. //

Another perspective on Def_V

A good question to ask now is:
how did anyone ever think of this MC functor construction?

The motivating example — via deformations of differentials — is one kind of answer but let me give another that fits well with the theme of DAG.

Other answer:

every dg Lie algebra L has an associated commutative dg algebra $C^*_{Lie}(L)$ and

$$MC_L(A) \cong CAlg^{aug}(C^*_{Lie}(L), A)$$

dg CLoc in Claudia's terminology

In other words, it is a version of dg commutative algebra (and hence algebraic geometry)

(This is an instance of "Koszul duality")

Rigel will describe this construction more carefully next week but let me sketch it now.

Consider the following double complex:

$$\underset{0}{\mathbb{K}} \xrightarrow{0} \underset{1}{L^*} \xrightarrow[\;d_{CE}\;]{[\,,\,]^*} \underset{2}{\Lambda^2 L^*} \longrightarrow \underset{3}{\Lambda^3 L^*} \longrightarrow \cdots \quad \underset{n}{\Lambda^n L^*} \quad \cdots$$

$[\ ]: \Lambda^2 L \to \mathbb{K}$ dualized

Take the product totalization:

$$\prod_n \Lambda^n(L^*)[-n] \cong \widehat{\mathrm{Sym}}(L^*[-1])$$

(completed symmetric algebra)

d_{CE} is dual of bracket extended as derivation:

$$d_{CE}(x\cdot y) = (d_{CE}x)\cdot y + (-1)^x x\cdot(d_{CE}y)$$

<u>Fact</u> (exercise) $d_{CE}^2 = 0$ follows from Jacobi identity!

We call ~~these~~ $C^*_{Lie}(L)$ the <u>Chevalley-Eilenberg cochains</u> of L.

What is a cdga map?

$$C^*_{Lie}(L) \xrightarrow{\varphi} A$$

Observation 1: φ is determined by where it sends generators

$$\varphi\big|_{Sym^1} : (L^*)[-1] \longrightarrow A$$

but A is in degree zero so

$$\varphi\big|_{Sym^1} : (L^*)^{-1} = (L^1)^* \longrightarrow A$$

and thus $\varphi \in Hom_{\mathbb{K}}((L^1)^*, A) \cong L^1 \otimes A$

cf the Maurer-Cartan construction

Claim 2: φ must be a map of cochain cplxs

In particular, φ must vanish on

$$d_{CE}(C^*(L)^{-1}) \subset C^*(L)^0$$

$\rightsquigarrow$ viewing $\varphi \in L^1 \otimes A$, this means

$$d_L \varphi + \frac{1}{2}[\varphi, \varphi] = 0$$

(apply the exercise about Jacobi!)